Dynamical Relativity in Family of Dynamics (Mathematical aspects of quantum fields and related topics)

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Dynamical Relativity in Family of Dynamics*

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1 Dynamical Relativity in Family of Dynamics

In this report, the discussion started in [1] is continued on "dynamical relativity" in "a family of dynamics" proposed recently by the author. The standard sorts of relativity like Einstein's are so formulated as to resolve the ***kinematical*** ambiguities caused by the unavoidable *non-uniqueness of reference frames* in theoretical descriptions of physical processes. In contrast, no systematic approaches seem to have been attempted so far to the problem of ***indeterminacy in dynamics*** caused by the presence of a *family of dynamics* or *constrained dynamics.* In the present discussion, the duality between the kinematical and dynamical relativities plays important roles, whose essence in an abstract categorical context can naturally be understood by the following duality [2] between **inductive** $\underset{\rightarrow}{Lim}$ & **projective** $\underset{\leftarrow}{Lim}$ limits:

$$[\text{Kinematics of } \underset{\rightarrow}{Lim} \overset{\swarrow}{\underset{\nwarrow}{\longleftarrow}}\ ] \overset{\text{duality}}{\leftrightarrows} [\ \overset{\nwarrow}{\underset{\swarrow}{\longleftarrow}} \underset{\leftarrow}{Lim} : \text{Dynamics in projective limit}]$$

due to the adjunctions involving the diagonal functor Δ [s.t. $\Delta(c)(j) \equiv c$ for $c \in \mathcal{C}$ $\forall j \in J$]:

$$\begin{array}{ccc} & \mathcal{C} & \\ \text{left adjoint} \quad \underset{\rightarrow}{Lim} \uparrow & \Delta \downarrow & \uparrow \underset{\leftarrow}{Lim} \quad \text{right adjoint} \\ & \mathcal{C}^J & \end{array}$$

To explain the essence, we need to clarify the following points:

1. Usual relativity principle as **kinematical** unification of **many reference frames** on sector-classifying space, such as Galileian relativity in non-relativistic physics, special relativity arising from electromagnetism due to Poincaré and Einstein, and Einstein's general relativity controlling gravity.

2. **Dynamical relativity** to unify dynamically a family of dynamics.

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3. **Duality** between kinematical & dynamical relativities:
   While "coordinate-free" nature of modern geometry is subsumed in Einstein's kinematical relativity, the plurality of ***indeterminate dynamics*** as the essence of dynamical relativity is dual to it, without being absorbed in the former one.

1.1 Sector-Classifying Space in Micro-Macro Duality

To clarify the meaning of a *sector-classifying space* in the above, we consider its roles in terms of the following basic concepts:

1) ***sectors*** as Micro-Macro boundary, which constitutes
2) ***Micro-Macro duality***, whose Macro side is formed through
3) ***emergence processes*** via "forcing" [Macro $\Longleftarrow$ Micro].

1.2 Sectors and Micro-Macro Duality

1) ***Sectors= pure phases*** parametrized by ***order parameters.***

Order parameters are the spectral values of central observables belonging to the centre $\mathfrak{Z}_\pi(\mathcal{X}) = \pi(\mathcal{X})'' \cap \pi(\mathcal{X})'$ of represented algebra $\pi(\mathcal{X})''$ of physical variables commuting with all other physical quantities in a generic representation π of $\mathcal{X}$. Mathematically, a ***sector*** is defined by a ***quasi-equivalence*** class of ***factor*** states (& representations π_γ) of the algebra $\mathcal{X}$ of physical quantities, characterized by ***trivial centre*** $\pi_\gamma(\mathcal{X})'' \cap \pi_\gamma(\mathcal{X})' = \mathfrak{Z}_{\pi_\gamma}(\mathcal{X}) = \mathbb{C}1$ as a *minimal unit* of representations classified by ***quasi-equivalence*** relation.

2) The roles of ***sectors as Micro-Macro boundary*** can be seen in ***Micro-Macro duality*** [3] as a mathematical version of "***quantum-classical correpsondence***" between the inside of microscopic ***sectors*** and the macroscopic ***inter-sectorial*** level described by geometrical structures on the central spectrum $Sp(\mathfrak{Z}) := Spec(\mathfrak{Z}_\pi(\mathcal{X}))$:

1.3 Micro-Macro Duality and Emergence of Macro-level

The situation can be conveniently described by a Hilbert bimodule ${}_{\pi(\mathcal{X})''}\widetilde{\mathcal{X}}_{L^\infty(E_\mathcal{X})} := \pi(\mathcal{X})'' \otimes L^\infty(E_\mathcal{X})$, with left $\pi(\mathcal{X})''$ action and right $L^\infty(E_\mathcal{X}, \mu)$ one (where $E_\mathcal{X}$ denotes the state space of $\mathcal{X}$ equipped with a central measure μ), controlled by the Tomita decomposition theorem:

⟵ Visible *Macro*	of	*independent objects*	$\cdots$	⟶	*Inter-sectorial*
$\cdots$ γ_N		***sectors*** γ	γ_2	γ_1	$Sp(3)$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\uparrow$ ***Intra-sectorial***
$\cdots$ π_{γ_N}		π_γ	π_{γ_2}	π_{γ_1}	$\Vert$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\downarrow$ Invisible ***Micro***

Then, Micro-Macro Duality is formulated as a categorical adjunction consisting of an adjoint pair of functors E, F together with a unit $\eta : I_{\mathcal{X}} \xrightarrow{\cdot} T$ and a counit $\varepsilon : S \xrightarrow{\cdot} I_{\mathcal{A}}$ intertwining, respectively, from $\mathcal{X}$ to the monad $T = EF$ and from the comonad $S = FE$ to $\mathcal{A}$:

emergence $\nearrow$	$\underset{\rightarrow}{Lim}$(frames): Spec $S = FE$ comonad $\searrow$	***kinematical relativity***
counit $\varepsilon \swarrow$	Arveson spec : $V \uparrow\downarrow I$: Spec subsp	$\searrow$ $\downarrow$: local net
States $\mathcal{A} \leftrightarrows$	bimodule of adjoint pair $\underset{E}{\overset{F}{\leftrightarrows}}$	$\mathcal{X}$ Algebra
dynamical relativity	$\uparrow\downarrow$: Galois	$\swarrow$ η: unit
	Dyn: monad $T = EF$ $\underset{\leftarrow}{Lim}$(dynamics)	$\nearrow$ co-emergence

Here the left adjoint functor F intertwines $FT = FEF = SF$ from monad T to comonad S and the right one E intertwines $ES = EFE = TE$ from S to T. The adjunction as natural isomorphisms $\mathcal{A}(a \leftarrow Fx) \underset{E(-)\eta_x}{\overset{\varepsilon_a F(-)}{\leftrightarrows}} \mathcal{X}(Ea \leftarrow x)$ is characterized by the two sets of identities $\begin{pmatrix} & FEF & \\ \varepsilon F \swarrow & \circlearrowleft & \nwarrow F\eta \\ F & = & F \end{pmatrix}$ and $\begin{pmatrix} E & = & E \\ E\varepsilon \nwarrow & \circlearrowright & \swarrow \eta E \\ & EFE & \end{pmatrix}$, as a homotopical extension of Fierz duality $E = F^{-1} \leftrightarrows F = E^{-1}$ between the orthgonality $FE = I_{\mathcal{A}}$ and the completeness $EF = I_{\mathcal{X}}$ of Fourier & inverse-Fourier transforms.

2 Galois-type Functors in *-categories

If the microscopic dynamics and the internal symmetry of the system are known from the outset, the principle of kinematical relativity tells us that observable quantities available in reality can essentially be specified as the invariants under the transformations of dynamics and symmetry. Since we do not live in the microscopic world, however, all what we can do is just to *guess* the *invisible* microscoipic dynamics and the internal symmetry on the basis of *visible* macroscopic data consisting of invariants under the transformations.

Therefore, the most essential tools in our scientific activities should be found in the methods to determine unknown quantities by solving such equations that the known coefficients are given in terms of observable invariants and that unobservable non-invariants are the unknown variables to be solved. For this reason, we need the basic concepts pertaining to the Galois theory of equations, among which the most important one is the Galois group. In the usual definition, a Galois group $G = Gal(\mathcal{X}/\mathcal{A}) =: G(\mathcal{X}, \mathcal{A})$ is defined by a pair of an algebra $\mathcal{X}$ containing knowns and unknowns, the former of which constitutes a subalgebra $\mathcal{A}$ of $\mathcal{X}$ providing coefficients of the equations, while the "quotient" $\mathcal{X}/\mathcal{A}$ has no actual meaning. If we interpret the symbol $/\mathcal{A}$ as $\mathcal{A}$ to be reduced to scalars, however, we can regard $\mathcal{X}/\mathcal{A}$ as a G-module whose inverse Fourier transform becomes $Gal(\mathcal{X}/\mathcal{A})$. With the aid of natural transformations, this re-interpretation can be extended categorically, according to which we obtain functors to extract groups or algebras from *-categories of modules as follows:

a) $G := End_{\otimes}(V : \mathcal{T}_{DR} \hookrightarrow FHilb)$: in Doplicher-Roberts sector theory [4], the group G of unbroken internal symmetry is recovered from the Doplicher-Roberts category $\mathcal{T}_{DR}(\subset End(\mathcal{A}))$ consisting of modules describing local excitations via the formula $G := End_{\otimes}(V : \mathcal{T}_{DR} \hookrightarrow FHilb)$ as the group of unitary $\otimes$-natural transformations u from the embedding functor V of $\mathcal{T}_{DR}$ into the category $FHilb$ of finite-dimensional Hilbert spaces

$$\text{to } V\text{:}\quad \begin{array}{ccc} V_{\gamma_1} & \overset{u(\gamma_1)=\gamma_1(u)}{\longleftarrow} & V_{\gamma_1} \\ T\downarrow & \circlearrowleft & \downarrow T \\ V_{\gamma_2} & \underset{u(\gamma_2)=\gamma_2(u)}{\longleftarrow} & V_{\gamma_2} \end{array} \quad \text{for } \gamma_i \in \mathcal{T}_{DR} \text{ and } T \in \mathcal{T}_{DR}(\gamma_2 \leftarrow \gamma_1) \text{ and}$$

$$\gamma_1(u) \otimes \gamma_2(u) = u(\gamma_1) \otimes u(\gamma_2) = u(\gamma_1 \otimes \gamma_2) = (\gamma_1 \otimes \gamma_2)(u).$$

b) $Nat(I : Mod_B \hookrightarrow Hilb) = B''$: Rieffel's device to extract the universal enveloping von Neumann algebra B'' from the category Mod_B of B-modules, in terms of natrual transformations from the embedding functor I to itself.

b') Takesaki-Bichteler's admissible family of operator fields on $Rep(B \to \mathfrak{H})$ in a sufficiently big Hilbert space $\mathfrak{H}$ to reproduce a von Neumann algebra B (: the example focused up in Dr. Okamura's PhD thesis as a non-commutative extension of Gel'fand-Naimark theorem).

With the aid of this machinery, such a perspective (as has long been advocated by Dr.Saigo and also emphasized recently by Dr.Okamura) can now be envisaged that all the contents of Quantum Field Theory can be unified into a C*-tensor category of physical quantities (joint work in progress).

3 Symmetry Breaking and Emergence of Sector-classifying Space

For discussing the third item 3) emergence processes via "forcing method" [Macro $\Longleftarrow$ Micro] to extract Macro from Micro, it is important to realize that the sector-classifying space typically emerges from spontaneous breakdown of symmetry of a dynamical system $\mathcal{X} \curvearrowleft G$ with action of a group G ("spontaneous" = without changing dynamics of the system). For this purpose, we need

Criterion for Symmetry Breaking given by non-triviality of ***central*** dynamical system $\mathfrak{Z}_\pi(\mathcal{X}) \curvearrowleft G$ arising from the original one $\mathcal{X} \curvearrowleft G$.

Namely, symmetry G is ***broken in sectors*** $\in Sp(\mathfrak{Z}) =: M$ ***shifted non-trivially by central action*** of G. In the infinitesimal version, the Lie algebra $\mathfrak{g}$ of the group G is decomposed into unbroken $\mathfrak{h}$ and broken $\mathfrak{m} := \mathfrak{g}/\mathfrak{h}$, the former of which is "vertical" to M and the latter "horizontal".

For the sector-classifying space M the assumption of its transitivity under the broken G leads to such a specific form as $M = G/H$ with H the ***unbroken*** subgroup. Then, the ***classical geometric*** structure on G/H can be seen to arise physically from an ***emergence*** process via ***condensation*** of a family of ***degenerate vacua***, each of which is mutually distinguished by condensed Macro values $\in Sp(\mathfrak{Z}) = M$ formed by infinite number of low-energy quanta.

In combination with the sector structure $\hat{H}$ of unbroken symmetry H, the total sector structure due to this symmetry breaking is described by a "sector bundle" $G \underset{H}{\times} \hat{H}$ with $\hat{H}$ as a standard fiber over a base space G/H of "***degenerate vacua***" [5, 6]. When this geometric structure is established, all the physical quantities are to be ***parametrized by condensed values*** $\in G/H$. Then, by means of "***logical extension***" of ***constants*** into ***sector-dependent variables***, we find the origin of local gauge structures. On these bases, the duality emerges between ***kinematical*** & ***dynamical*** sorts of "relativity principles" owing to the duality between converging & diverging families of functors between Macro & Micro: [Kinematics in $\underrightarrow{Lim}$ $\overset{\swarrow}{\underset{\nwarrow}{\longleftarrow}}$] $\overset{\text{duality}}{\leftrightarrows}$ [$\overset{\nwarrow}{\underset{\swarrow}{\longleftarrow}}$ $\underleftarrow{Lim}$: Dyn in projective limit].

3.1 Symmetric Space Structure of G/H

We see here that this homogeneous space $M = G/H$ is a ***symmetric space*** equipped with Cartan involution as follows (IO, in preparation). Assuming Lie structures on $G, H, G/H = M$, we have the corresponding Lie algebraic quantities denoted, respectively, by $\mathfrak{g}, \mathfrak{h}, \mathfrak{m}$, satisfying $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$. Then the validity of $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ provides the homogeneous space M (at least, locally) with a Cartan involution $\mathcal{I}$ to characterize a symmetric space whose eigenvalues are $\mathcal{I} = +1$ on $\mathfrak{h}$ and $\mathcal{I} = -1$ on $\mathfrak{m}$, respectively. Note that $[\mathfrak{m}, \mathfrak{m}]$ is the ***holonomy*** term corresponding to an infinitesimal loop along the ***broken direction*** $G/H = M = Sp(3)$ as ***inter-sectorial space***. Namely, $[\mathfrak{m}, \mathfrak{m}]$ describes the effect of ***broken*** G transformation along an infinitesimal loop on M starting from a point in M and going back to the same point. According to the above Criterion for Symmetry Breaking in terms of ***non-trivial shift under central action*** of G, the absence of $\mathfrak{m}$-components in $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, follows from the identity of initial and final points of the loop. Thus, $M = G/H = Sp(3)$ is a symmetric space.

3.2 Example 1: Relativity controlled by Lorentz group

Typical example of the above sort can be found in the case of Lorentz group $\mathcal{L}_+^\uparrow =: G$ with an unbroken subgroup of the rotation group $SO(3) =: H$: here, $G/H = M \cong \mathbb{R}^3$ is a symmetric space of Lorentz frames mutually connected by Lorentz boosts.

With $\mathfrak{h} := \{M_{ij}; i, j = 1, 2, 3, i < j\}$, $\mathfrak{m} := \{M_{0i}; i = 1, 2, 3\}$, the validity of $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ is evident from the basic Lie algebra structure:

$$[iM_{\mu\nu}, iM_{\rho\sigma}] = -(\eta_{\nu\rho} iM_{\mu\sigma} - \eta_{\nu\sigma} iM_{\mu\rho} - \eta_{\mu\rho} iM_{\nu\sigma} + \eta_{\mu\sigma} iM_{\nu\rho}).$$

While both $\mathfrak{h}$ and $\mathfrak{m}$ are taken as unbroken in the standard physics, such results as Borcher-Arveson theorem (: affiliation of Poincaré generators to the algebra of global observables in vacuum situation) and the spontaneous breakdown of Lorentz boosts at $T \neq 0K$ [7] indicate the ***speciality of the vacuum situation with*** $\mathfrak{m}$ ***unbroken***. In this sense, the symmetric space of Lorentz frames $M \cong \mathbb{R}^3$ with [boosts, boosts] = rotations, gives a typical example of symmetric space structure emerging from symmetry breaking (inevitable in non-vacuum situations).

Along this line, typical examples are provided by the chiral symmetry with the current algebra structure $[V, V] = V, [V, A] = A, [A, A] = V$ with vector currents V and axial vector ones A, and also by the conformal symmetry. In the latter case consisting of translations P_μ, Lorentz transformations $M_{\mu\nu}$, scale transformation S and of special conformal transformations K_μ the unbroken $\mathfrak{h}$ part corresponds to $M_{\mu\nu}$ and S, and the broken $\mathfrak{m}$ to P_μ and K_μ, where $\mathfrak{m}$ is the infinitesimal non-compact form of the self-dual Grassmannian manifold acted by the conformal group.

3.3 Example 2: Second law of thermodynamics

Physically most interesting example can be found in thermodynamics: corresponding to $\mathfrak{h} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{m} = \mathfrak{g}/\mathfrak{h}$, we find here an exact sequence $\Delta' Q \hookrightarrow \Delta E = \Delta' Q + \Delta' W \twoheadrightarrow \Delta' W$ due to the first law of thermodynamics, whose precise form can be found in Caratheodory's formulation. With respect to Cartan involution with + assigned to the heat production $\Delta' Q$ and $-$ to the macroscopic work $\Delta' W$, the holonomy $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ corresponding to a loop in the space M of thermodynamic variables becomes just

Kelvin's version of second law of thermodynamics,

namely, holonomy $[\mathfrak{m}, \mathfrak{m}]$ in the cyclic process with $\Delta E = \Delta' Q + \Delta' W = 0$, describes the heat production $\Delta' Q \geq 0$: $-\Delta' W = -[\mathfrak{m}, \mathfrak{m}] = \Delta' Q > 0$ (from the system to the outside).

Thus, the essence of the second law of thermodynamics is closely related with the geometry of the symmetric space structure of thermodynamic space M consisting of paths of thermodynamic state-changes caused by works $\Delta' W$. Actually, this symmetric space structure can be seen to correspond to its *causal structure* due to state changes via adiabatic processes, which can be interpreted as the mathematical basis of Lieb-Yngvason axiomatics of thermodynamic entropy.

4 Kinematics vs. Dynamics: Kinematical Convergence at Macro End

In relation with symmetric space structure, an essential feature of kinematical convergence on the Macro side can be seen in the basic structure of relativity similar to thermodynamcs. Because of this, phenomenological diversity due to ***many reference frames*** is successfully controlled by the relativity principle with the aid of Lorentz-type transformations. In this situation, however, the roles played by the implicit assumption should not be overlooked about the ***unicity of "true physical system"*** in such a form as the unique microscopic law of dynamics in sharp contrast to the phenomenological diversity. But who guarantees its validity? This point should be contrasted with the universal validity of thermodynamic consequences applicable to variety of different systems independently of minor details. From the duality viewpoint between Micro and Macro: [Macro: $\varinjlim \overset{\swarrow}{\underset{\nwarrow}{\longleftarrow}}$] $\overset{\text{duality}}{\leftrightarrows}$ [$\overset{\nwarrow}{\underset{\swarrow}{\longleftarrow}} \varprojlim$: Micro], mentioned at the beginning, we should notice the one-sidedness inherent in the standard picture of relativity: [Kinematics in $\varinjlim \overset{\swarrow}{\underset{\nwarrow}{\longleftarrow}}$], in contrast to the situations on the Micro side: [$\overset{\nwarrow}{\underset{\swarrow}{\longleftarrow}} \varprojlim$: Dyn in projective limit]. As a typical example of such one-sidedness, we note here the incompatibility between the requirement of relativistic covariance and the presence of interactions among constituents for any finite systems of relativistic particles carrying timelike energy-momentum $p = (p_\mu)$, $p^2 \geq 0$; this no-go theorem can naturally be understood on the basis of a sharp dichotomy in the energy-momentum supports of field operators between those restricted in $p^2 \geq 0$ for ***free*** fields and the one extending all over $p \in \widehat{\mathbb{R}^4}$ for interacting Heisenberg fields, owing to famous Haag's no-go theorem.

5 Renormalization: Duality between "Cutoffs" to Circumvent Haag's No-Go Theorem

Haag's no-go theorem mentioned above means the disjointness (=absence of non-zero intertwiners) between interacting Heiseberg fields φ_H and the corresponding asymptotic ***free*** fields $\phi^{in/out}$, due to the mismatch between their p-space support properties: $[\mathrm{supp}(\widetilde{\varphi_H}(p)) = \widehat{\mathbb{R}^4}]$ ***vs.*** $[\mathrm{supp}(\widetilde{\phi^{in/out}(p)}) \subset V_+ \cup (-V_+)]$. Such a sharp result follows from the complex analyticity due to the basic universal postulates imposed on the relativistic quantum field theory (QFT, for shor), in such a form as the spectrum condition, owing to which any holes in p-space support inevitably eliminate interactions (as long as spacetime covariance is preserved).

In the standard perturbative approach to QFT, a "cutoff" is introduced to regularize the ultraviolet divergences appearing in the Feynman diagrams. Because of its artificial appearance, this procedure is regarded as a "necessary evil" to be avoided as much as possible, in preference for the renormalization procedure to recover formally the relativistic covariance by "removing" the explicit form of cutoffs. In relation with Haag's no-go theorem, however, an essential role played by the "cutoff" should properly be noticed in circumventing the inconvenient and inevitable consequence of this theorem: namely, without the breakdown of relativistic covariance due to the "cutoff", any access to the interacting theory is impossible starting from the available theory consisting of free fields. In view of the difference in scales validating the physical meaning of spacetime points and of quantum fields, however, it is groundless to believe in the existence of a universal procedure to justify any field operators defined at a spacetime point, independently of the choice of scales to be discussed. For instance, if a certain class of states with moderate energy contents are selected by imposing such a condition as "energy bounds", any field polynomials can safely be evaluated at a point in such states (see below). This should be contrasted to the p-space integral summing up all the energy momentum of internal lines with equal weight, which is the origin of the familiar ultraviolet divergences. From the viewpoint of non-unique choices of possible different cutoffs, we find the unavoidable ambiguity in the consistent treatment of the dynamics on the Micro side of QFT, which will necessitate the idea of "family of dynamics" to be accepted, as in the case of degenerate dynamics in gauge theory. Thus, similarly to gauge sectors corresponding to gauge fixing conditions, the renormalization sectors parametrized by variables dual to "cutoffs" are expected to appear, which are mutually connected by scale transformations as renormalization-group transformations. In the following, we try to summarize the relevant materials for justifying this scenario.

5.1 Nuclearity condition & OPE

As a specific form of "cutoff" to circumvent Haag's theorem, we choose here in vacuum Hilbert space $\mathfrak{H}$ a subspace $\mathfrak{H}_{\mathcal{O},E} := \{P_E\mathcal{X}(\mathcal{O})\Omega\}^-$ localized in a finite spacetime domain $\mathcal{O}$ carrying energy-momentum $p_\mu \in V_+$ with energy $\leq E$, by means of spectral projection P_E.

According to the *nuclearity condition* postulated in algebraic QFT, a subset $\{P_E\mathcal{X}(\mathcal{O})_1\Omega\}^-$ in $\mathfrak{H}_{\mathcal{O},E}$ corresponding to the unit ball $\mathcal{X}(\mathcal{O})_1$ in the observable (or field) algebra $\mathcal{X}(\mathcal{O})$ is a *nuclear set*, admitting such a decomposition as

$$\Phi_{\mathcal{O},E}(A) = \sum_{i=1}^{\infty} \varphi_i(A)\xi_i \quad \text{for } \forall A \in \mathcal{A}(\mathcal{O})_1$$

$$\text{with } \varphi_i \in \mathcal{A}(\mathcal{O})^* \text{ and } \xi_i \in \mathfrak{H} \text{ s.t. } \sum_{i=1}^{\infty} ||\varphi_i||\,||\xi_i|| < \infty,$$

On this setting-up, the operator-product expansion (OPE) is shown to be valid non-perturbatively (Bostelmann '00) as follows:

5.2 Non-perturbative OPE and normal operators

For localized states $\omega \in E_{\mathcal{X}(\mathcal{O})}$ with mild energy-momentum dependence characterized by the "energy bounds" condition $\omega((1+H)^n) < \infty$, a *field* $\hat{\phi}(x)$ *at a point* x can safely be defined (BOR '01).

However, *their products at a point* x being meaningless should be replaced by *normal products*: e.g., ill-defined square $\hat{\phi}(x)^2$ is replaced by a linear space $\mathcal{N}(\hat{\phi}^2)_{q,x}$ of normal products $\hat{\Phi}_j(x)$, $j = 1, \cdots, J(q)$, appearing in the following OPE:

$$||(1+H)^{-n}\left[\hat{\phi}(x+\frac{\xi}{2})\hat{\phi}(x-\frac{\xi}{2}) - \sum_i \hat{\Phi}_i(x)C_i(\xi)\right](1+H)^{-n}||$$
$$\leq c\,|\xi|^q,$$

which is valid for spacelike $\xi (\in \mathbb{R}^4) \to 0$ with arbitrary $q > 0$, by choosing a finite number of fields $\hat{\Phi}_j(x)$ and sufficiently large n, and some analytic functions $\xi \mapsto c_j(\xi)$, $j = 1, \cdots, J(q)$.

5.3 Counter terms

Singularity of product $\hat{\phi}(x+\frac{\xi}{2})\hat{\phi}(x-\frac{\xi}{2})$ *in the limit of* $\xi \to 0$ is isolated into kinematical c-number factors $C_i(\xi) = N_i(\lambda)C_i^{reg}(\xi)$, where $\lambda := |\xi|^{-1}$ is *cut-off momentum* to regularize *UV divergences* in a *non-perturbative* way and

$N_i(\lambda)$ can be taken as *counter terms* to define *renormalized field operators* by

$$\hat{\phi}_{ren}(x) := \Pi_i N_i(\lambda)^{-1/2}\hat{\phi}(x).$$

(1) *Counter terms* $N_i(\lambda)$ are expected to be *factors of automorphy* associated to fractional linear transformations of (approximate) *conformal* symmetry $SO(2,4)(\simeq SU(2,2))$ following from (approximate) scale invariance. Along this line, Callan-Symanzik type equation for $N_i(\lambda)$ involving running coupling constants and anomalous dimensions should be established.

5.4 Nuclearity condition as renormalizability

(2) *renormalizability* = finite number of types of "1-particle irreducible (1PI)" divergent diagrams is expected to follow from nuclearity condition (= *intra-sectorial* structure);

In this sense, *nuclearity condition* can be regarded as *mathematical version of renormalizability condition* and *broken scale invariance* inherent to local subalgebras $\mathcal{A}(\mathcal{O})$ of *type III* with no minimal projection requires *renormalization condition to be specified at some renormalization point* which can, however, be *chosen arbitrarily.*

(3) *absence of minimal projection in type III* von Neumann factors (due to approximate scale invariance) allows *shifts of renormalization points* by scale transformations = *renormalization-group transformations.* This gives *inter-sectorial* relations among *"sectors parametrized by renormalization conditions" at different renormalization points* (on the centre $\mathfrak{Z}(\widehat{\mathcal{A}}) = \mathfrak{Z}(\widehat{\mathcal{A}}(\mathcal{O})) = C(\mathbb{R}^+)$ of scaling algebra).

5.5 For further developments

(a) *In the opposite direction* to the conventional renormalization scheme based on *perturbative expansion method* starting from a *"Lagrangian"* (along such a flow chart as "Lagrangian" $\to$ perturbative expansion $\to$ renormalization + OPE), perturbation expansion itself should be derived and justified as a kind of asymptotic analysis within the non-perturbative formulation of renormalization based on OPE: namely, we advocate such a flow chart as starting from OPE $\to$ renormalization $\to$ perturbative method as asymptotic expansion $\to$ "Lagrangian" determined by Γ_{1PI} & renormalizability (= finite generation property).

(b) More detailed mathematical connections should be clarified among *nuclearity condition, renormalizability, renormalization conditions, renormalization group to shift renormalization point* and *broken scale invariance inherent to local subalgebras* $\mathcal{A}(\mathcal{O})$ *of type III* from the viewpoint of *nonstandard analysis.*

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